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ON THE SOLUTION OF A CLASS OF WIENER-HOPF INTEGRAL EQUATIONS IN FINITE AND INFINITE RANGES

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by

Raj Mittra

15 May 1959

Contract No. AF33(616)-6079

Project No. 9-(13-6278) Task 40572

Sponsored by:

WRIGHT AIR DEVELOPMENT CENTER



ELECTRICAL ENGINEERING RESEARCH LABORATORY
ENGINEERING EXPERIMENT STATION
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1. INTRODUCTION

The subject of investigation in this report is the Wiener-Hopf Integral Equation

$$g(t) = \int_0^T f(\tau) K|t-\tau| d\tau$$

for a T either finite or infinite. It is assumed that $K|t|$ admits of an expression of the form

$$K|t| = \sum_m C_m e^{-k_r |t|}$$

The method of solution for the semi-infinite range ($T = \infty$) is discussed first. The method is then extended to T finite. Two basically different approaches are used. The first one leads to a set of simultaneous equations which can be solved by analytic means.

The second one uses the technique of Fourier transforms in a complex plane and is more suitable for an iterative method of solution for the finite T case (for $T = \infty$, the Wiener-Hopf technique yields the answer directly).

The eigenvalue problem corresponding to the above kernel is also discussed. Suitable numerical examples are included and certain applications in the fields of Information Theory and Electromagnetic Wave propagation are also pointed out.

2. SOLUTION OF THE SEMI-INFINITE RANGE INTEGRAL EQUATION

Consider the integral equation

$$g(t) = \int_0^{\infty} f(\tau) K |t-\tau| d\tau \quad t > 0 \quad (1)$$

where $K |t-\tau|$ is of the type

$$K |t-\tau| = \sum_{r=1}^n C_r e^{-k_r |t-\tau|}, \quad \text{Re}(k_r) > 0 \quad (2)$$

C_r = constant coefficients

Equation (1) is the well known Wiener-Hopf Integral Equation and a solution of the equation can be obtained for a general $K(t)$ by the use of the method of Integral Transforms in the complex plane¹. For the particular type of Kernel which appears in (2), however, the equation can be solved directly by using the theory of residues of a complex variable. The first step for solving the problem is similar to the one used by Laning and Battin². A set of simultaneous equations are obtained when their method is used and these equations are solved here using the technique of residues.

2.1 Derivation of the Simultaneous Set of Equations

The function $g(t)$ can be expanded in a series of exponentials

$$g(t) = \sum_r D_r e^{-l_r t}, \quad t > 0$$

Hence the problem in Equation (1) can be reduced to that of solving for a $g(t)$ of the form $e^{-l t}$ and superposing the contribution due to the different terms in the series.

An alternative approach is the following. Operate on both sides of (1) with the operator L , where

$$L = \prod_{r=1}^n (p^2 - k_r^2), \quad p = \frac{d}{dt}$$



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Then, using

$$(p^2 - k_r^2) e^{-k_r |t-\tau|} = -2k_r \delta(t-\tau), \quad (\delta(t) = \text{Dirac delta})$$

one obtains,

$$L g(t) = \prod_{g=1}^n (p^2 - k_g^2) g(t) = -2 \sum_{r=1}^n k_r C_r \prod_{m=1}^n (p^2 - k_m^2) f(t) \quad (3)$$

where the superscript (n) in the continued product implies that the factor corresponding to $m = n$ is to be omitted. Equation (3) is the differential equation satisfied by $f(t)$. The equation is clearly inhomogeneous and can be written in the form

$$\frac{F_1(p^2)}{F_2(p^2)} \cdot f(t) = g(t) \quad 0 < t < \infty \quad (4)$$

It can be easily shown that $F_1(-p^2)/F_2(-p^2)$ is the Fourier transform of the kernel function $K(t)$, i.e.

$$\int_{-\infty}^{\infty} K(t) e^{-ipt} dt = \frac{F_1(-p^2)}{F_2(-p^2)}$$

Returning to (4), one knows that (4) has a particular solution say $f_p(t)$, and a general* solution of the integral equation can be written in the form

$$f(t) = f_p(t) + A_0 \delta(t) + \sum_{i=1}^{n-1} A_i e^{-a_i t}, \quad t > 0 \quad (5)$$

where a_i 's are the roots of $F_1(p^2)$. From (3) one sees that the order of the polynomial F_1 in p^2 is $(n-1)$, so $F_1(p^2)$ has symmetrical roots,

$$p = \pm a_1, \pm a_2 + \dots \pm a_{n-1}$$

* For the case, the degree of $F_1(p^2)$ is one less than that of $F_2(p^2)$. When the difference in the degrees is more than one, the method discussed in the next section is applicable.

Only half of these roots, however, the ones with the negative sign, will give solutions with exponential decay as $t \rightarrow +\infty$. The others, with a positive real part in the exponent are not admissible as solutions. Also it can be shown (see for instance Reference 2, p. 416, that the solution of the integral equation must contain the Dirac- δ included in (5).

Let $g_1(t)$, $0 < t < \infty$, be defined as

$$\begin{aligned} g_1(t) &= \int_0^\infty f_p(\tau) K(|t-\tau|) d\tau \\ &= \int_0^\infty \left\{ f(\tau) - A_0 \delta(\tau) - \sum_{i=1}^{n-1} A_i e^{-\alpha_i \tau} \right\} K(|t-\tau|) d\tau \end{aligned} \quad (6)$$

using (1) and (2), (6) can be written as

$$\begin{aligned} g_1(t) - g(t) &= \int_0^\infty (\sum_i A_i e^{-\alpha_i \tau}) (\sum_r C_r e^{-k_r |t-\tau|}) d\tau \\ &= A_0 \sum_r C_r e^{-k_r t} \quad t > 0 \end{aligned} \quad (7)$$

Upon evaluation of the integral in the r.h.s. of (7), the following is obtained,

$$g_1(t) - g(t) = \sum_i D_i e^{-k_i t}$$

where D_i 's can be calculated in terms of A_0 , A_i 's and the C_r 's.

The coefficients of $e^{-\alpha_i t}$ can be shown to be equal to zero since α_i 's are the roots of $F_1(p^2)$. Hence the right hand side contains terms involving the exponentials $e^{-k_i t}$ only.

The left hand side of (8) is known if the particular solution $f_p(t)$ and hence $g_1(t)$ is known. From (8), remembering that the coefficients of $e^{-\alpha_i t}$ are zero, it follows that

$$g_1(t) - g(t) = \sum_i C_i P_i e^{-k_i t} \quad (9)$$

where P_i^l 's can be obtained from the knowledge of $g_1(t)$.

Equating the coefficients of $e^{-k_r t}$ the following set of n - equations are obtained for the n -unknowns $A_0, A_1 \dots A_{n-1}$,

$$A_0 + \sum_{j=1}^{n-1} \frac{A_j}{a_j - k_r} = P_r, \quad r = 1, 2, \dots, n \quad (10)$$

where P_r^l 's in Equation (10) are considered as known.

The solution of (10) will now be derived using the theory of residues.

2.2 Solution of the Set of Equation for the Unknown Coefficients A_j

Consider first of all the solution of a set of equations

$$A_0^{(m)} + \sum_{j=1}^{n-1} \frac{A_j^{(m)}}{a_j - k_r} = P_m \delta_r^m, \quad m = 1, 2, \dots, n \quad (11)$$

If $A_0^{(m)}$ and $A_j^{(m)}$'s can be solved for all ' m 's, then $A_0, A_1, \dots$ etc are obtained by superposition, i.e., using

$$A_0 = \sum_{m=1}^n A_0^{(m)}, \quad A_j = \sum_{m=1}^n A_j^{(m)} \quad (12)$$

Attention will therefore be devoted presently to the solution of (11)

For this purpose consider a contour integral of the form,

$$\oint_C \frac{Q(z)}{z - k_r} dz,$$

where $Q(z)$ satisfies the following conditions,

1. $\lim_{z \rightarrow \infty} Q(z) = Q_0$ (constant)
2. $Q(z)$ has $(n-1)$ poles at $z = a_1, a_2 \dots a_{n-1}$.
3. $Q(z)$ has $(n-1)$ zeros at $z = k_r, r = 1, 2, m-1, m+1, \dots, n$, excluding $n = m$

Such a choice of $Q(z)$ is readily found as,

$$Q(z) = Q_0 \frac{\prod_{r=1}^n (z - k_r)^{(m)}}{\prod_{r=1}^{n-1} (z - a_r)}, \quad (Q_0 = \text{constant}) \quad (13)$$

Now let the contour in the integral

$$\oint_C \frac{Q(z)}{z-k_r} dz$$

be chosen large enough to include all the singularities of $Q(z)$. If one lets the contour C recede to infinity, say by taking an infinitely large circle for C , one has:

$$\oint_C Q_0 \frac{Q(z)}{z-k_r} dz = 2\pi j Q_0 = 2\pi j \sum_{p=1}^{n-1} \frac{Q(z)}{z-k_r} (z-a_p) \Big|_{z=a_p} + 2\pi j Q(z) \Big|_{z=k_r} \quad (14)$$

Since $Q(z)$ has zeros for $z = k_r$, $r \neq m$, (14) can be written as

$$Q_0 - \sum_{p=1}^{n-1} \frac{Q(z)(z-a_p)}{(z-k_r)} \Big|_{z=a_p} = Q(z) \Big|_{z=k_m} \delta_n^m \quad (15)$$

Using the expression of $Q(z)$ given in (13) in the Equation (15), the following results,

$$Q_0 - \sum_{p=1}^{n-1} Q_0 \left\{ \frac{\prod_{r=1}^n (a_p - k_r)}{\prod_{r=1}^{n-1} (a_p - a_r)} \right\} \cdot \frac{1}{a_p - k_r} = Q_0 \left\{ \frac{\prod_{r=1}^n (k_m - k_r)}{\prod_{r=1}^{n-1} (k_m - a_r)} \right\} \delta_r^m \quad (16)$$

Comparing (16) with (11), it is seen that if one puts the following relations in the equation (11),

$$Q_0 \left\{ \frac{\prod_{r=1}^n (k_m - k_r)}{\prod_{r=1}^{n-1} (k_m - a_r)} \right\} = P_m \quad (17a)$$

$$Q_0 = A_0^{(m)} \quad (17b)$$

$$-Q_0 \left\{ \frac{\prod_{r=1}^n (a_p - k_r)}{\prod_{r=1}^{n-1} (a_p - a_r)} \right\} = A_p^{(m)} \quad (17c)$$

The set of 'n' equations are identically satisfied. Q_0 in (17) can be calculated from the knowledge of P_m . The unknowns in (11), viz $A_0^{(m)}$, $A_1^{(m)}$ $A_n^{(m)}$ etc., are then obtained using (17b) and (17c).

Since the a_p 's and k_r 's are known, the solution of (11) is complete. Also, as was pointed out earlier, the solution of (10) can be constructed from the solutions of (11) with the use of (12). The solution of the Integral Equation (1) is then obtained by substituting the coefficients A_0 , and A_1 in (5).

2.3 Extension to the Case When the Difference of Order of $F_1(p^2)$ and $F_2(p^2)$ is more than one

In the previous section the derivation of a simultaneous set of equations for A_0 , A_1 ... etc. was based on the assumption that the order of the polynomial $F_1(p^2)$ is one less than that of $F_2(p^2)$. In this section this condition will be relaxed and the process for formulating a new set of equations will be outlined. The method of solution will also be presented.

Let, 'g' and 'n' be the order of $F_1(p^2)$ and $F_2(p^2)$ respectively where $g < n$. As has been pointed out by Laning and Battin (Ref 2.2 p. 416), the solution $f(t)$ will now contain the derivatives of the Dirac Delta and $f(t)$ will be of the form

$$f(t) = A_0 \delta(t) + \sum_{i=1}^{s-1} (-1)^i A_i \delta^{(i)}(t) + \sum_{i=1}^q A_i e^{-a_i t} \quad t > 0 \quad (18)$$

$$S = n - q > 1, \quad \delta^{(i)} = i\text{-th derivative of } \delta$$

Following exactly the same method as was used for the derivation of (10) one now obtains the set of equations,

$$A_0 + \sum_{i=1}^{s-1} A_i k_r^i + \sum_{i=1}^q \frac{A_i'}{a_i - k_r} = P_r \quad (19)$$

where P_r 's are the same as in (10).

Again consider solving the equation similar to (11), viz ,

$$A_o^{(m)} + \sum_{i=1}^{s-1} A_i^{(m)} k_r^i + \sum_{i=1}^q \frac{A_i'^{(m)}}{a_i^{-k_r}} = p_m \delta_r^m \quad (20)$$

by using the method of residues. Let $M(z)$ be a function of the complex variable z , which satisfies

$$1. \quad \lim_{z \rightarrow \infty} \left\{ M(z) - \sum_{i=1}^{s-1} L_i z^i \right\} \rightarrow L_o \quad (\text{constant})$$

2. $M(z)$ has q poles at $z = a_i$, $i = 1, 2, \dots, q$

3. $M(z)$ has $(n-1)$ zeros at $z = k_r$, $r = 1, 2, \dots, n$, $r \neq m$

A choice of $M(z)$ which satisfies the above conditions is

$$M(z) = M_o \left\{ \frac{\prod_{i=1}^n (z - k_i)^{(m)}}{\prod_{i=1}^q (z - a_i)} \right\} \quad (21)$$

One can find the constants L_i and L_o in terms of M_o , k_r and a_i from the equation (21).

Now consider the integral

$$I = \oint_C \left\{ \frac{M(z)}{(z - k_r)} - \sum_{i=1}^{s-1} \frac{L_i z^i}{(z - k_r)} \right\} dz \quad (22)$$

where C is an infinitely large circle.

Since the integrand behaves as L_o/z as $z \rightarrow \infty$, one has

$$I = 2\pi j L_o = 2\pi j \left\{ \sum_{i=1}^q \left. \frac{M(z)}{z - k_r} (z - a_i) \right|_{z=a_i} - \sum_{i=1}^{s-1} L_i k_r^i + M(z) \right|_{z=k_m} \delta_r^m \right\} \quad (23)$$

$r = 1, 2, \dots, n$

Equation (23) can be rewritten as

$$L_o + \sum_{i=1}^{s-1} L_i k_r^i - \sum_{i=1}^q \left\{ \left. M(z) (z - a_i) \right|_{z=a_i} \right\} \frac{1}{a_i^{-k_r}} = M(z) \bigg|_{z=k_m} \delta_r^m \quad (24)$$

Hence, comparing the r.h.s of (24) and (20) and using the expression for $M(z)$ in (21), one sets $M(z) \Big|_{z=k_m} = P_m$, i.e.

$$M_0 = P_m \left\{ \frac{\prod_{i=1}^q (k_m - a_i)}{\prod_{i=1}^{n^{(m)}} (k_m - k_i)} \right\} \quad (25)$$

As was explained above the constants $L_0, L_1, \dots, L_{(s-1)}$ are determined once M_0 is known.

Again comparing (24) with (20) one has,

$$L_0 = A_0^{(m)}, \quad L_i = A_i^{(m)}, \quad i = 1, 2, \dots, (s-1)$$

$$-M(z)(z - a_r) \Big|_{z=a_r} = M_0 \left\{ \frac{\prod_{i=1}^{n^{(m)}} (a_r - k_i)}{\prod_{i=1}^q (a_r - a_i)} \right\} A_r'^{(m)} \quad (26)$$

$$r = 1, 2, \dots, q$$

The expression (26) yields all the unknowns in (20). The solution of (19) is obtained by using superposition. The method of solution has thus been extended in this section for the case when the degree of the polynomial in $F_1(p^2)$ differs by more than one from the degree of $F_2(p^2)$. Recall here that $F_1(-p^2)/F_2(-p^2)$ is the Fourier Transform of the kernel $K|t|$.

3. THE INTEGRAL EQUATION WITH A FINITE RANGE

The solution of the integral equation,

$$g(t) = \int_0^T f(\tau) K|t-\tau| d\tau \quad (27)$$

which is similar to (1) except for the upper limit, will be detailed in this section. $K|t|$ is again assumed to be expressible as a series of exponentials $e^{-k_i|t|}$ as in (2).

The method formulating the problem in terms of a set of simultaneous equations, is similar to the one used in the previous section. The solution of these equations, however, is considerably more involved.

The Differential Equation for $f(t)$ and the particular solution $f_p(t)$ are the same as obtained above in (24), if of course $g(t)$ and $K(t)$ are the same. The range however is different, i.e.

$$\frac{F_1(p^2)}{F_2(p^2)} f(t) = g(t) \quad 0 < t < \tau \quad (28)$$

Hence a general solution of (28) is of the form

$$f(t) = f_p(t) + \sum_{i=1}^{n-1} A_i e^{-a_i t} + \sum_{i=1}^{n-1} B_i e^{a_i(t-\tau)} \quad (29)$$

where a_i 's are the same as defined in the previous section.

It is assumed that the order of $F_1(p^2)$ is one less than that of $F_2(p^2)$. The method can be extended for the case when this difference is more than one, as was discussed before. It can also be shown that the solution of the Integral Equation must contain Dirac - δ 's i.e. the complete expression for $f(t)$ which satisfies (27) is,

$$f(t) = f_p(t) + A_0 \delta(t) + B_0 \delta(t-T) + \sum_{i=1}^{n-1} A_i e^{-a_i t} + \sum_{i=1}^{n-1} B_i e^{a_i(t-T)} \quad (30)$$

The problem in hand is to find the $2n$ coefficients, A_0 , B_0 , A_i 's and B_i 's.

One again defines a $g_1(t)$ as

$$g_1(t) = \int_0^T f_p(\tau) K|t-\tau| d\tau$$

and it is possible to verify, by following the lines used in deriving (29) that

$$g_1(t) - g(t) = \sum_{R=1}^n C_R P_R e^{-k_R t} + \sum_{r=1}^n C_R R_r e^{k_r(t-T)} \quad (31)$$

Substituting (30) in (27) one obtains after carrying out the integration (for details see Laning, *ibid.* p. 320)

$$\begin{aligned} \int_0^T \left\{ f(\tau) - f_p(\tau) \right\} K|t-\tau| d\tau &= \int_0^T \left\{ \sum_{i=1}^{n-1} A_i e^{-a_i \tau} + \sum_{i=1}^{n-1} B_i e^{a_i(t-T)} \right\} K|t-\tau| d\tau \\ &= \sum_r \left(A_0 + \sum_{i=1}^{n-1} \frac{A_i}{a_i - k_r} - \sum_i \frac{B_i e^{-a_i T}}{a_i + k_r} \right) C_r e^{-k_r t} \\ &\quad + \sum_r \left(B_0 + \sum_i \frac{B_i}{a_i - k_r} - \sum_{i=1}^{n-1} \frac{A_i e^{-a_i T}}{a_i + k_r} \right) C_r e^{k_r(t-T)} \end{aligned} \quad (32)$$

Since (31) and (32) should be identical for all values of (t) , one gets the following set of equations by equating the coefficients of $e^{-k_r t}$ and $e^{k_r(t-T)}$.

$$A_0 + \sum_{i=1}^{n-1} \frac{A_i}{a_i - k_r} - \sum_{i=1}^{n-1} \frac{B_i e^{-a_i T}}{a_i + k_r} = P_r \quad (33a)$$

$$B_0 + \sum_{i=1}^{n-1} \frac{B_i}{a_i - k_r} - \sum_{i=1}^{n-1} \frac{A_i e^{-a_i T}}{a_i + k_r} = Q_r \quad (33b)$$

$$r = 1, 2, \dots, n$$

(33a) and (33b) are the desired set of $2n$ equations for the unknowns A_o, A_i, B_o, B_i .

It is to be noted that (33) reduces to the form (10) when $T \rightarrow \infty$.

3.1 Solution of the Simultaneous Set of Equations Obtained for the Finite Range Case

The solution of the set of $2n$ equations for A_o, A_i, B_o , and B_i , are discussed in this section. As a first step to solving (33), form a new set of equations by adding and subtracting (33a) and (33b) and obtaining

$$F_o + \sum_i F_i \left(\frac{1}{a_i - k_r} - \frac{e^{-a_i T}}{a_i + k_r} \right) = P_r + Q_r \quad (34a)$$

$$J_o + \sum_i J_i \left(\frac{1}{a_i - k_r} - \frac{e^{-a_i T}}{a_i + k_r} \right) = P_r - Q_r \quad (34b)$$

where

$$F_o = A_o + B_o, \quad J_o = A_o - B_o$$

$$F_i = A_i + B_i, \quad J_i = A_i - B_i$$

The advantage of using (34a) and (34b) in preference to (33a) and (33b) becomes immediately clear if one notices that the set (34a) and (34b) are uncoupled. This is because the matrix corresponding to (34a) is the same as that of (34b). Hence, it is only necessary to invert the $(n \times n)$ matrix corresponding to one set say (34a), rather than the $2n \times 2n$ matrix of (33a) and (33b).

The inversion of the matrix (34a) is not straight forward, however, since the residue method is not directly applicable.

It is nevertheless possible to construct the elements of the inverted matrix for (34a), when the cofactors and the determinant of the matrix for the set (10) is known. This is detailed in the following section.

3.2 Inversion of the Matrix of the Equation (34a)

The matrix $[M']$ for the equation (34a) is,

$$[M'] = \begin{bmatrix} 1, \left(\frac{1}{a_1 - k_1} - \frac{e^{-a_1 T}}{a_1 + k_1} \right), \left(\frac{1}{a_2 - k_1} - \frac{e^{-a_2 T}}{a_2 + k_1} \right) \cdots \left(\frac{1}{a_{n-1} - k_1} - \frac{e^{-a_{n-1} T}}{a_{n-1} + k_1} \right) \\ 1, \left(\frac{1}{a_1 - k_2} - \frac{e^{-a_1 T}}{a_1 + k_2} \right), \left(\frac{1}{a_2 - k_2} - \frac{e^{-a_2 T}}{a_2 + k_2} \right) \cdots \left(\frac{1}{a_{n-1} - k_2} - \frac{e^{-a_{n-1} T}}{a_{n-1} + k_2} \right) \\ \vdots \\ 1, \left(\frac{1}{a_1 - k_n} - \frac{e^{-a_1 T}}{a_1 + k_n} \right) \cdots \left(\frac{1}{a_{n-1} - k_n} - \frac{e^{-a_{n-1} T}}{a_{n-1} + k_n} \right) \end{bmatrix} \quad (35)$$

The matrix corresponding to (10), say $[M]$ is

$$[M] = \begin{bmatrix} 1, \frac{1}{a_1 - k_1}, & \frac{1}{a_r - k_r}, & \cdots, & \frac{1}{a_{n-1} - k_1} \\ 1, \frac{1}{a_1 - k_r}, & \frac{1}{a_r - k_r}, & \cdots, & \frac{1}{a_{n-1} - k_r} \\ \vdots & \vdots & \ddots & \vdots \\ 1, \frac{1}{a_1 - k_n}, & \frac{1}{a_r - k_n}, & \cdots, & \frac{1}{a_{n-1} - k_n} \end{bmatrix} \quad (36)$$

Let the cofactors of the determinant of the matrix $[M]$ be designated by the symbol Δ_{pq} .

Then

$$\Delta_{pq} = \Delta_{pq} (a_1, a_2, \dots, a_{q-2}, a_q, \dots, a_{n-1}, k_1, k_2, \dots, k_{p-1}, k_{p+1}, \dots, k_n) \quad (37)$$

i.e. the expression for Δ_{pq} involves all a_i 's and k_r 's excluding a_{q-1} and k_p and this can be easily verified from the form of (36).

The determinant Δ for $[M]$ however involves all the a_i 's and k_r 's.

Suppose now Δ_{pq} is known, then, Δ'_{pq} corresponding to $[M']$ can be expressed in terms Δ_{pq} , as follows,

$$\begin{aligned}
 \Delta'_{pq} = & \Delta_{pq} + \sum_{i_1=1}^{n-1} e^{-a_{i_1} T} \Delta_{pq}(-a_{i_1}) \\
 & + \sum'_{i_1=1}^{n-2} \sum'_{i_2=i_1+1}^{n-1} e^{-(a_{i_1}+a_{i_2})T} \Delta_{pq}(-a_{i_1}, -a_{i_2}) \\
 & + \sum'_{i_1=1}^{n-3} \sum'_{i_2=i_1+1}^{n-2} \sum'_{i_3=i_2+1}^{n-1} e^{-(a_{i_1}+a_{i_2}+a_{i_3})T} \Delta_{pq}(-a_{i_1}, -a_{i_2}, -a_{i_3}) \\
 & + \dots\dots\dots \\
 & + e^{-(a_{i_1}+a_{i_2}+\dots+a_{i_{n-1}})T} \Delta_{pq}(-a_{i_1}, -a_{i_2}, \dots, -a_{i_{n-1}})
 \end{aligned} \tag{38}$$

where the prime on the summation symbol implies that the term corresponding to $(q-1)$ is to be omitted.

Also, $\Delta_{pq}(-a_{i_1}, -a_{i_2}, \dots)$ implies that the signs of the particular set of a 's appearing in the argument are to be changed in the expression for Δ_{pq} keeping the signs of the other a 's unchanged. As an example, if

$$\Delta_{pq} = (k_2 - k_3)(k_1 - k_2)(a_1 - a_2)(a_2 - a_3)$$

then

$$\Delta_{pq}(-a_1) = (k_2 - k_3)(k_1 - k_2)(-a_1 - a_2)(a_2 - a_3)$$

and

$$\Delta_{pq}(-a_1, -a_3) = (k_2 - k_3)(k_1 - k_2)(-a_1 - a_2)(a_2 + a_3),$$

and so on.

Equation (38) is a rather long and involved one, but it is a straightforward process to show its validity. Furthermore, in a practical problem it is only necessary to compute a few of the terms in the series (38). This is because the higher order terms have an exponential decay with the order.

Equation (38) provides a means for obtaining the cofactors of the matrix $[M']$ when the cofactor of the matrix $[M]$ is known. The cofactors of $[M]$ can be obtained by a method outlined in the next section.

In order to find the inverse of the matrix $[M']$, one still needs the expression for the determinant Δ' of $[M']$ in addition to the expression for the cofactors. The expression is very similar to that of Δ'_{pq} and is given in the following.

$$\begin{aligned}
 \Delta' = \Delta &+ \sum_{i_1=1}^{n-1} e^{-a_{i_1} T} \Delta(-a_{i_1}) \\
 &+ \sum_{i_1=1}^{n-2} \sum_{i_2=i_1+1}^{n-1} e^{-(a_{i_1}+a_{i_2})T} \Delta(-a_{i_1}, -a_{i_2}) \\
 &+ \dots \\
 &+ e^{-(a_{i_1}+a_{i_2}+\dots+a_{i_{n-1}})T} \Delta(-a_{i_1}, -a_{i_2}, \dots, -a_{i_{n-1}})
 \end{aligned} \tag{39}$$

where the symbols have the same meaning as used in connection with (38).

Using (38) and (39) it is possible to construct the element N_{pq} , which is p, qth element of the inverse of $[M']$ by using the well known expression

$$N_{qp} = \frac{\Delta'_{pq}}{\Delta'}$$

The problem of inverting the matrix $[M']$ and solving the set of Equations (34a) and (34b) can thus formally be considered as solved.

It has been tacitly assumed in the above that Δ'_{pq} and Δ' can be obtained by some reasonably simple means. The actual method for calculating them is discussed in the following section.

4. CALCULATION OF THE DETERMINANT AND THE COFACTORS OF THE DETERMINANT OF THE MATRIX M

In the previous chapter it was shown that the elements of the matrix $[M]^{-1}$ can be developed from the knowledge of the determinant and the cofactors of the determinant of $[M]$. The matrix $[M]$ is associated with the set of Equation (10).

Consider first of all the task of expressing the determinant Δ of $[M]$ in a suitable form. The determinant is rewritten here for convenience

$$\Delta = \begin{vmatrix} 1 & \frac{1}{(a_1 - k_1)} & \cdots & \frac{1}{(a_{n-1} - k_1)} \\ 1 & \frac{1}{(a_1 - k_2)} & \cdots & \frac{1}{(a_{n-1} - k_2)} \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & \frac{1}{(a_1 - k_n)} & \cdots & \frac{1}{(a_{n-1} - k_n)} \end{vmatrix} \quad (39)$$

The determinant can be easily transformed to the form of a double alternant (for the definition of an alternant see Reference 3), by subtracting the first row from the rest. After this operation, when the common factors are taken out, there results,

$$\Delta = \begin{vmatrix} 1 & \frac{1}{(a_1 - k_1)} & \frac{1}{(a_2 - k_1)} & \cdots & \frac{1}{(a_{n-1} - k_1)} \\ 0 & \frac{1}{(a_1 - k_2)} & \frac{1}{(a_2 - k_2)} & \cdots & \frac{1}{(a_{n-1} - k_1)} \\ 0 & \vdots & & & \\ 0 & \vdots & & & \\ 0 & \vdots & & & \\ 0 & \frac{1}{(a_1 - k_n)} & \frac{1}{(a_2 - k_n)} & \cdots & \frac{1}{(a_{n-1} - k_n)} \end{vmatrix} \quad \times F \quad (40)$$

where

$$F = \frac{\prod_{r=2}^n (k_r - k_1)}{n-1 \prod_{r=1}^n (a_r - k_1)}$$

Now consider the $(n-1)$ th order determinant V where

$$V = \begin{vmatrix} \frac{1}{(a_1 - k_2)} & \frac{1}{(a_2 - k_2)} & \cdots & \frac{1}{(a_{n-1} - k_2)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{a_1 - k_n} & \frac{1}{(a_2 - k_n)} & \cdots & \frac{1}{(a_{n-1} - k_n)} \end{vmatrix} \quad (41)$$

The determinant V is a double alternant and can be conveniently expressed as a ratio of products as follows

$$V = \frac{(-1)^{n-1} \zeta^{\frac{1}{2}}(a_1, a_2, \dots, a_{n-1}) \cdot \zeta^{\frac{1}{2}}(-k_2, -k_3, \dots, -k_n)}{\prod_{n=1, m=2}^{n-1, n} (a_r - k_m)} \quad (42)$$

where $\zeta^{\frac{1}{2}}$ is a difference product defined as

$$\zeta^{\frac{1}{2}}(x_1, x_2, \dots, x_q) = \prod_{k=1, \ell=2}^{\ell-1, q} (x_k - x_\ell) \quad (43)$$

From (40)

$$\begin{aligned} \Delta &= V \cdot F \\ &= \frac{\zeta^{\frac{1}{2}}(a_1, a_2, \dots, a_{n-1}) \zeta^{\frac{1}{2}}(-k_1, -k_2, \dots, -k_n)}{\prod_{r=1, m=1}^{n-1, n} (a_r - k_m)} \end{aligned} \quad (44)$$

Equation (44) gives the desired expression for the determinant Δ of the matrix $[M]$. The expression for the cofactors are also determined in a similar manner.

Consider first of all the cofactors Δ_{m1} , where

$$\Delta_{m1} = (-1)^{m+1} \begin{vmatrix} \frac{1}{a_{11} - k_1} & \cdots & \frac{1}{a_{n-1,1} - k_1} \\ \vdots & & \vdots \\ \frac{1}{a_{1,m-1} - k_{m-1}} & \cdots & \frac{1}{a_{n-1,m-1} - k_{m-1}} \\ \vdots & & \vdots \\ \frac{1}{a_{1,m+1} - k_{m+1}} & \cdots & \frac{1}{a_{n-1,m+1} - k_{m+1}} \\ \vdots & & \vdots \\ \frac{1}{a_{1,n} - k_n} & \cdots & \frac{1}{a_{n-1,n} - k_n} \end{vmatrix} \quad (45)$$

The determinant in expression (45) is a double alternant. Hence, Δ_{m1} can be expressed as

$$\Delta_{m1} = (-1)^{m+1} \frac{\zeta^{\frac{1}{2}}(a_1, a_2, \dots, a_{n-1}) \zeta^{\frac{1}{2}}(-k_1, -k_2, \dots, -k_n)^{(m)}}{\prod_{\substack{r=1, p=1 \\ p \neq m}}^{n-1, n} (a_r - k_p)} \quad (46)$$

where the superscript on the second product implies that k_m is omitted from the argument.

Next consider Δ_{mq} , $q \neq 1$

$$\Delta_{mq} = (-1)^{m+q} \begin{vmatrix} \frac{1}{a_{11} - k_1} & \cdots & \frac{1}{a_{q-1,1} - k_1} & \frac{1}{a_{q+1,1} - k_1} & \cdots & \frac{1}{a_{n-1,1} - k_1} \\ \vdots & & \vdots & & & \vdots \\ \frac{1}{a_{1,m-1} - k_{m-1}} & \cdots & \frac{1}{a_{q-1,m-1} - k_{m-1}} & \frac{1}{a_{q+1,m-1} - k_{m-1}} & \cdots & \frac{1}{a_{n-1,m-1} - k_{m-1}} \\ \vdots & & \vdots & & & \vdots \\ \frac{1}{a_{1,m+1} - k_{m+1}} & \cdots & \frac{1}{a_{q-1,m+1} - k_{m+1}} & \frac{1}{a_{q+1,m+1} - k_{m+1}} & \cdots & \frac{1}{a_{n-1,m+1} - k_{m+1}} \\ \vdots & & \vdots & & & \vdots \\ \frac{1}{(a_{1,n} - k_n)} & \cdots & & & & \frac{1}{a_{n-1,n} - k_n} \end{vmatrix} \quad (47)$$

The determinant appearing in (47) is of the type Δ , and the method used above for expressing Δ is applicable here. The first step is to subtract the first row from the others and take out the common factors. The resulting determinant is a double alternant and can be expressed conveniently.

Omitting the details, the final expression for Δ_{mq} , $q \neq 1$ is given below

$$\Delta_{mq} = (-1)^{m+q} \frac{\zeta^{\frac{1}{2}}(a_1, a_2, \dots, a_{n-1})^{(q)} \zeta^{\frac{1}{2}}(-k_1, -k_2, \dots, -k_n)^{(m)}}{\prod_{\substack{\tilde{n}=1, \tilde{n} \\ r=1, \ell=1 \\ r \neq q, \ell \neq m}} (a_r - k_\ell)} \quad (48)$$

Equations (44), (46) and (48) are the desired expressions for the determinant and the cofactors of the determinant Δ of the matrix $[M]$.

5. THE FOURIER TRANSFORM METHOD OF FORMULATION

In the previous chapter the method of solving the equation,

$$g(t) = \int_0^T f(t) K(t-\tau) d\tau \quad 0 < t < T$$

$g(t)$ specified for $0 < t < T$

involved as a first step, the formulation in terms of a set of simultaneous equations. This was possible because the kernel of the Integral Equation could be expressed in a suitable series of exponentials. In this section there is discussed an alternative method of formulating the problem using the technique of Fourier Transformation in the complex plane. The advantage of this method is that the formulation is useful for a general $K|t|$, the only requirement being that the existence of the transform for the function $K|t|$. It is also assumed that the asymptotic behavior of $g(t)$ is such that the transform can be defined when $g(t)$ is extrapolated beyond the range it is specified.

It is also demonstrated that when the kernel $K(t-\tau)$ can be expressed in the form of a series of exponentials, as was the case assumed before, the simultaneous set of equations can also be derived from the present formulation. The exact solution is then possible by the method described in the previous section. For a general kernel $K(t-\tau)$, however, the solution is obtained approximately by a series of iterations. An insight into this approach is gained by first solving the particular case of the series form of $K(t-\tau)$. This is pointed out in the discussion given in connection with the iterative method of solution.

5.1 Fourier Transform Formulation

The formulation in terms of the Fourier Transforms is discussed in the following.

The Fourier Transform $H(\omega)$ of a function $h(t)$ is usually defined as

$$H(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} h(t) dt, \quad \omega \text{ real.}$$

The inverse transform of $H(\omega)$ is then given by,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} H(\omega) d\omega$$

For the purpose of this work, however, the transforms will be defined in terms of a complex variable $s = s_1 + is_2$.

The definitions of Fourier Transform of $h(t)$ in the complex plane which are suitable here¹ are given below.

Let $h(t)$ be a function such that $f(t) e^{-a_1 t}$, $a_1 > 0$ is in class L^2 in the interval $0 < t < \infty$, and $f(t) e^{-a_2 t}$, $a_2 < 0$, is in L^2 for $-\infty < t < 0$, then

$$F_+(s) = \int_0^{\infty} f(t) e^{ist} dt, \quad s = s_1 + is_2, \quad s_2 > a_1 \quad (49)$$

and

$$F_-(s) = \int_{-\infty}^0 f(t) e^{ist} dt, \quad s_2 < a_2 \quad (50)$$

The inversion formulas are

$$\frac{1}{2\pi} \int_{-\infty + ia_1}^{\infty + ia_1} F_+(s) e^{-ist} ds = f(t), \quad t > 0 \quad (51)$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty + ia_2}^{\infty + ia_2} F_-(s) e^{ist} ds &= f(t) \quad t < 0 \\ &= 0, \quad t > 0 \end{aligned}$$

Hence the Fourier Transform for $h(t)$ is given by

$$\text{F.T. } [h(t)] = H_+(s) + H_-(s)$$

Now consider the Fourier Transform of a function $f_1(t)$ defined as

$$\begin{aligned} f_1(t) &= f(t) & 0 < t < T \\ &= 0 & \text{otherwise} \end{aligned} \quad (52)$$

In general $f(t)$ can be expressed as

$$f(t) = q(t) + r(t) \quad 0 < t < T$$

where

$$q(t) = \sum A_n e^{-c_n t} \quad \operatorname{Re}(c_n) \geq 0 \quad (53)$$

$$r(t) = \sum B_n e^{d_n(t-T)} \quad \operatorname{Re}(d_n) \geq 0 \quad (54)$$

Note that $q(t)$ and $r(t)$ decrease exponentially for ' t ' positive and negative respectively.

The transform for $f_1(t)$, $F_1(s)$ can be expressed as

$$\begin{aligned} F_1(s) &= \text{F.T. of } f_1(t) \\ &= \int_{-\infty}^{\infty} e^{-ist} f_1(t) dt = \int_0^T \{q(t) + r(t)\} e^{-ist} dt \end{aligned} \quad (55)$$

Rewriting (53) one has

$$\begin{aligned} F_1(s) &= \int_0^{\infty} q(t) e^{-ist} dt - \int_T^{\infty} q(t) e^{-ist} dt \\ &\quad + \int_{-\infty}^T r(t) e^{-ist} dt - \int_{-\infty}^0 r(t) e^{-ist} dt \end{aligned} \quad (56)$$

Using the shift theorem for transforms it is possible to write,

$$\int_T^\infty q(t) e^{-ist} dt = e^{-isT} V_+(s) \quad (57)$$

where

$$V_+(s) = \int_0^\infty q(t+T) e^{-ist} dt$$

similarly

$$\int_{-\infty}^T r(t) e^{-ist} dt = e^{-isT} W_-(s) \quad (58)$$

$$W_-(s) = \int_{-\infty}^0 r(t+T) e^{-ist} dt$$

Also let

$$Q_+(s) = \int_0^\infty q(t) e^{-ist} dt, \quad (59)$$

and

$$R_-(s) = \int_{-\infty}^0 r(t) e^{-ist} dt \quad (60)$$

Putting (57), (58), (59) and (60) in (56), there results,

$$F_1(s) = Q_+(s) - e^{-isT} V_+(s) - e^{-isT} W_-(s) - R_-(s) \quad (61)$$

Note that the integral in (55) was broken up into (56) because of the behavior of $q(t)$ and $r(t)$ given in (53) and (54). The range was changed from $0 < t < T$ to $0 < t < \infty$ and $T < t < \infty$, so that the transforms could be defined in terms $Q_+(s)$, $V_+(s)$ and $W_-(s)$, $R_-(s)$ which are analytic

in the complex s -plane in the ranges shown below.

$$Q_+(s) \text{ and } V_+(s) \text{ analytic for } \operatorname{Im}(s) > 0$$

$$W_-(s) \text{ and } R_-(s) \text{ analytic for } \operatorname{Im}(s) < 0$$

Now consider the transform of a function $g(t)$ which is known in the range $0 < t < T$

Then let

$$g(t) = g_1(t) + g_2(t) \quad 0 < t < T \quad (62)$$

where say

$$g_1(t) = \sum P_n e^{-\nu_n t} \quad \operatorname{Re}(\nu_n) \geq 0 \quad (63)$$

and

$$g_2(t) = \sum Q_n e^{\delta_n(t-T)}, \quad \operatorname{Re}(\delta_n) \geq 0 \quad (64)$$

Also let $g(t)$ be extrapolated for all t in the form,

$$g(t) = g_1(t) U(t) + g_2(t) U(T-t) + g_3(t) U(t-T) + g_4(t) U(-t) \quad (65)$$

In (65) the unknowns are $g_3(t)$ and $g_4(t)$ and $U(t)$ = unit step, i.e.

$$\begin{aligned} U(t) &= 0 & t < 0 \\ &= 1 & t > 0 \end{aligned}$$

Let the F.T. of $g(t)$ be $G(s)$, then

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} e^{-ist} g(t) dt = \int_0^{\infty} e^{-ist} g(t) dt \\ &\quad + \int_{-\infty}^0 g(t) e^{-ist} dt \end{aligned} \quad (66)$$

Using (65) and (66), the following is obtained by obvious application

of the shift theorem.

$$G(s) = G_{1+}(s) + e^{+isT} I_{1-}(s) + e^{-isT} I_{2+}(s) + G_{4-}(s) \quad (67)$$

where

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ist} g_2(t) U(T-t) dt &= \int_{-\infty}^T e^{-ist} g_2(t) dt \\ &= e^{-isT} \int_{-\infty}^0 e^{-ist} g_2(t+T) dt = e^{-isT} I_{1-}(s) \end{aligned} \quad (68)$$

and similarly

$$I_{2+}(s) = \int_0^{\infty} e^{-ist} g_3(t+T) dt.$$

Also

$$\text{F.T.} \left[g_1(t) U(t) \right] = G_{1+}(s)$$

and

$$\text{F.T.} \left[g_4(t) U(-t) \right] = G_{4-}(s).$$

Also if $K(t) \rightarrow 0$ as $|t| \rightarrow \infty$, then the transform of $K(t)$ say $\mathcal{K}(s)$ is,

$$\mathcal{K}(s) = \int_{-\infty}^{\infty} K|t| e^{-ist} dt \quad (69)$$

The equations (61), (67) and (69) give the transforms of $f(t)$, $g(t)$ and $K(|t|)$ respectively under the assumptions mentioned.

If now the F.T. is taken of the original Integral Equation,

$$g(t) = \int_0^T f(\tau) k|t-\tau| d\tau \quad 0 < t < T$$

by first writing it in the form

$$g_1(t) U(t) + g_2(t) U(T-t) + g_3(t) U(t-T) + g_4(t) U(-t)$$

$$= \int_{-\infty}^{\infty} f_1(t) k |t-\tau| d\tau$$

One gets using the expressions for the appropriate transforms developed above,

$$\begin{aligned} & G_{1+}(s) + G_{4-}(s) + e^{-isT} \left\{ I_{1-}(s) + I_{2+}(s) \right\} \\ &= \left[Q_+(s) - R_-(s) + e^{-isT} \left\{ W_-(s) - V_+(s) \right\} \right] K(s) \end{aligned} \quad (70)$$

The unknowns in (70) are $G_{4-}(s)$, $I_{2+}(s)$, $Q_+(s)$, $R_-(s)$, $W_-(s)$ and $V_+(s)$. Regrouping terms in (70) there results,

$$\begin{aligned} & G_{1+}(s) + G_{4-}(s) - Q_+(s) - R_-(s) \mathcal{K}(s) \\ &= e^{-isT} \left[\left\{ W_-(s) - V_+(s) \right\} K(s) - I_{1-}(s) - I_{2+}(s) \right] \end{aligned} \quad (71)$$

From the definition of the various transforms, it is possible to verify that the only terms dependent on T are the ones appearing on the r.h.s. of (71). Since (71) is to be valid for all T , the left and right sides of (71) must separately be equal to zero. The two equations which are obtained using this condition are,

$$\left\{ Q_+(s) - R_-(s) \right\} \mathcal{K}(s) = G_{1+}(s) + G_{4-}(s) \quad (72)$$

and

$$\left\{ W_-(s) - V_+(s) \right\} \mathcal{K}(s) = I_{1-}(s) + I_{2+}(s) \quad (73)$$

The known quantities in (72) are $\mathcal{K}(s)$ and $G_{1+}(s)$ and in (73), they are $\mathcal{K}(s)$ and $I_{1-}(s)$.

The equations can be solved if there exists a common strip of analyticity of the transformed functions involved.

5.2 Development of a Set of Simultaneous Equations

Consider the case when there exists an expansion of $K(t)$ in the form

of a series

$$K(|t|) = \sum_{r=1}^M C_r e^{-k_r |t|} \quad \text{Re}(k_r) > 0$$

and when

$$g(t) = \sum_{r=1}^n C_r P_r e^{-k_r t} + \sum_{r=1}^n C_r Q_r e^{k_r (t-T)}$$

It was shown in Chapter 1, that such a $g(t)$ is obtained when the particular solution $f_p(t)$ is filtered out of the integral equation by calculating

$$\int_0^T \left\{ f(\tau) - f_p(\tau) \right\} K|t-\tau| d\tau$$

Then from (62),

$$g_1(t) = \sum_{r=1}^n C_r P_r e^{-k_r t}$$

and

$$g_2(t) = \sum_{r=1}^n C_r Q_r e^{k_r (t-T)}$$

Using

$$\int_0^{\infty} e^{-k_r t} e^{-ist} dt = \frac{1}{k_r + is}$$

one has the expressions for $G_{1+}(s)$ and $I_{1-}(s)$, as,

$$G_{1+}(s) = \sum \frac{C_r P_r}{k_r + is} \quad (74)$$

and

$$I_{1-}(s) = \sum \frac{C_r Q_r e^{-k_r T}}{k_r - is} \quad (75)$$

$$\text{F.T. } (K|t|) = \mathcal{K}(s) = 2 \sum \frac{C_r k_r}{k_r^2 + s^2} = d \frac{\prod_{r=1}^{n-1} (s^2 + a_r^2)}{\prod_{r=1}^{n-1} (s^2 + k_r^2)}$$

where d is a constant and a_r 's are the zeros of $\mathcal{K}(s)$. Equation (72) and (73) for this case becomes

$$\left\{ Q_+(s) - R_-(s) \right\} \cdot d \frac{\prod_{r=1}^{n-1} (s^2 + a_r^2)}{\prod_{r=1}^{n-1} (s^2 + k_r^2)} = \sum \frac{C_r P_r}{k_r - is} + G_{4-}(s) \quad (76)$$

and

$$\left\{ W_-(s) - V_+(s) \right\} \cdot d \frac{\prod_{r=1}^{n-1} (s^2 + a_r^2)}{\prod_{r=1}^{n-1} (s^2 + k_r^2)} = \sum \frac{C_r Q_r}{k_r + is} e^{-k_r T} + I_{2+}(s) \quad (77)$$

From the nature of the differential equation of which $f_p(t)$ is the particular solution one knows that the complementary solution $f(t) - f_p(t)$ has the form,

$$A_0 \delta(t) + \sum_{r=1}^{n-1} A_r e^{-a_r T} + B_0 \delta(t-T) + \sum_{i=1}^{n-1} B_i e^{-a_i (t-T)}$$

Therefore $Q_+(s)$ can be assumed to be of the form

$$Q_+(s) = A_0 + \sum \frac{A_r}{a_r - is} \quad (78)$$

Similarly

$$W_-(s) = B_0 + \sum \frac{B_r}{a_r + is} \quad (79)$$

The unknowns are A_0 , A_i 's, B_0 and B_i 's.

It is possible to express the remaining quantities in the l.h.s. of (76) and (77), i.e. $R_-(s)$ and $V_+(s)$, in terms of the coefficients

A'_s and B'_s . For this purpose refer to (57) and (58) where the definitions of $R_-(s)$ and $V_+(s)$ were given. It is seen that

$$\text{F.T.}^{-1} R_-(s) = r(t) U(-t) \quad (80)$$

and

$$\text{F.T.}^{-1} W_-(s) = r(t+T) U(-t) \quad (81)$$

If $R_-(s)$ is chosen to be

$$R_-(s) = \sum_r \frac{B_r e^{-a_i T}}{a_r + is} \quad (82)$$

The Equations (80) and (81) are consistent as is verified by inverting $R_-(s)$, and using (79).

Similarly

$$\text{F.T.}^{-1} V_+(s) = q(t+T)$$

and

$$\text{F.T.}^{-1} Q_+(s) = q(t)$$

Hence

$$V_+(s) = A_o + \sum_r \frac{A_r e^{-a_i T}}{a_r - is} \quad (83)$$

Substituting (78), (79), (80) and (81) in (76) and (77), one gets after dividing both sides by $\mathcal{K}(s)$

$$\left\{ A_o + \sum_r \frac{A_r}{a_r - is} - \sum_r \frac{B_r e^{-a_i T}}{a_r + is} \right\} = \frac{1}{\mathcal{K}(s)} \sum_r \frac{C_r P_r}{k_r - is} + \frac{G_{4-}(s)}{\mathcal{K}(s)} \quad (84)$$

and

$$\left\{ B_o + \sum_r \frac{B_r}{a_r + is} - \sum_r \frac{A_r e^{-a_i T}}{a_r - is} \right\} = \frac{1}{\mathcal{K}(s)} \sum_r \frac{C_r Q_r e^{-k_r T}}{k_r + is} + \frac{I_{2+}(s)}{\mathcal{K}(s)} \quad (85)$$

Notice that $1/\mathcal{K}(s)$ has zeros for $i_s = \pm k_r$. Hence if one sets $i_s = k_m$ in (84), $m = 1, 2, \dots, n$, one obtains the set of equations,

$$A_o + \sum \frac{A_r}{a_{r-k_m}} - \sum \frac{B_r e^{-a_r T}}{a_{r+k_m}} = \frac{1}{\{(k_m - i_s)\} \mathcal{K}(s)} \bigg|_{s=ik_m} \quad m=1, 2, \dots n \quad (86)$$

This is because all the terms in the summation on the right hand side go to zero except the one containing $1/k_m - i_s$ in the denominator. Notice that $G_{4-}(s)$ can not have a pole for $s = -ik_m$ since it is regular in the lower half of the s - plane.

Similarly, from (37) one obtains by setting $i_s = -k_m$ or $s = ik_m$,

$$B_o + \sum \frac{B_r}{a_{r-k_m}} - \sum \frac{A_r e^{-a_r T}}{a_{r+k_m}} = \frac{C_{mQ_m} e^{-k_m T}}{[\mathcal{K}(s)(k_m + i_s)]} \bigg|_{s=ik_m} \quad m = 1, 2, \dots n \quad (87)$$

By comparison one notices that (86) and (87) have the same form as (33a) and (33b) in Chapter 3. In particular the matrix associated with these equations is exactly the same. Methods discussed in Chapter 3 can therefore be used to solve (86) and (87).

5.3 Iterative Method of Solution

It was shown in the discussion above that the transformed equations (72) and (73) can be reformulated in terms of a set of simultaneous equations similar to the ones obtained in Chapter 3. This was shown to be possible when the kernel $K(t-\tau)$ is expressible in a suitable series of exponentials. However, this might not be a convenient thing to do for all kernels. It is therefore useful to find alternative methods for solving (72) and (73). An iterative method for solving the above equations is presented here.

The method of solution is based on the following reasoning. Suppose one starts with an approximation of $q(t)$, say $q^{(1)}(t)$. Let $q^{(1)}(t)$ be the solution of the problem for $T = \infty$. $q^{(1)}(t)$ is therefore obtainable by using Wiener-Hopf method. The following steps are carried out for the iterative procedure.

1. Predict $q^{(1)}(t+T) = \text{approximate } q(t+T) \text{ using } q^{(1)}(t) \text{ which is the solution of the equation corresponding to } T = \infty.$

2. Find $V^{(1)}_+(s)$, the transform of $q^{(1)}(t+T)$, substitute in (73) and solve for $W^{(1)}_-(s)$, the transform of $r(t+T)$ (see the definitions of V_+ and W_- in (57) and (58), by using Wiener-Hopf technique.
3. Find $V^{(1)}_-(t)$ by inverting $W^{(1)}_-(s)$ and substitute in (73). Next solve for $q^{(2)}(t)$ by applying Wiener-Hopf method to (72).
4. Repeat steps (1), (2) and (3) until the solution converges.

No proof is included here regarding the convergence of the process. However, it is possible to make the following observations. If $q(t+T)$ is small for $t \geq 0$, i.e. if the solution $q(t)$ decays with increasing t and if T is large enough to make $q(t+T)$ small, then the solution $r_1(t+T)$ is a good approximation of the actual $r(t)$. Also if $r(t)$ is fairly small for $t \leq 0$, then from the definition of $R_-(s)$ it is again seen that $q_1(t)$ obtained from (72) is a close approximation of $q(t)$. The process is then expected to converge fast.

Although it might not be practically convenient to express $K|t-\tau|$ in a series of exponentials, it is nevertheless useful for making certain statements about the convergence of the iterative process by considering a formal series expansion of $K|t-\tau|$ in terms of exponentials. From the discussion in Chapter 4 (see Equation (42)) that

$$f(t) - f_p(t) = A_0 \delta(t) + \sum_{i=1} A_i e^{a_i t} + B_0 + \sum_{i=1} B_i e^{a_i (t-T)}$$

From the above expression it is seen that $q(t+T)$ for $t \geq T$ and $r(t)$ for $t \leq 0$ are small provided T is large or the a_i 's are sufficiently large. It is therefore observed that if the kernel has a reasonably fast decay with the increase of its argument, and the particular solution is filtered out from $f(t)$ in the beginning, it is expected that the convergence of the iterative process will also be reasonably fast.

It is not necessary that the kernel be of an exponentially decaying type. The essential condition for a rapid convergence of the process is that $q(t)$ and $r(t-T)$ tend asymptotically to the corresponding semi-infinite range solution at $t = T$. It is assumed in the above that the particular solution has been filtered out from the general one. Considerations such

as these apply to kernels of the type

$$H_0^{(2)} \left\{ k \left| t - \tau \right| \right\}$$

which appear in the formulation of electro-magnetic wave problems, say for instance in the problem of diffraction of e.m. waves by a perfectly conducting strip.

6. THE EIGENVALUE PROBLEM

In the preceding chapters the discussion was centered around solving the integral equation

$$g(t) = \int_0^T f(\tau) K(|t-\tau|) d\tau \quad 0 < t < T$$

$g(t)$ specified in the range $0 < t < T$ and methods were presented for solving the equation when

$$K|t| = \sum_{r=1}^n C_r e^{-k_r |t|}$$

In a class of problems in probability theory the eigenvalue problem

$$\phi(t) = \lambda \int_0^T \phi(\tau) K|t-\tau| d\tau$$

is encountered. The origin of the Equation (88) and a transform method of solution has been discussed by Youla⁴ in a recent paper. The purpose here is to show that the eigenvalue problem can be formulated in terms of a matrix without resorting to the transform method. The direct method of formulation is simple and involves comparatively less amount of work.

6.1 The Eigenvalue Matrix Equation

Assume $\phi(t)$ in the form

$$\phi(t) = \sum_{m=1}^n (A_m e^{-\gamma_m t} + B_m e^{\gamma_m t}) \quad 0 < t < T \quad (89)$$

Such a form is dictated by the fact that the integral

$$\int_0^T e^{-\gamma_m \tau} K|t-\tau| d\tau$$

yields $e^{-\gamma_m t}$ multiplied by a factor, plus some other terms of the type $e^{-k_r t}$ and $e^{k_r t}$. If by adjusting A_m 's and B_m 's, the coefficients of

$e^{-k_r t}$ and $e^{k_r t}$ in the integrated expression of

$$\int_0^T \phi(\tau) K |t-\tau| d\tau,$$

can be made equal to zero, then it will be possible to obtain the solution of the eigenvalue problem.

When (89) is substituted in (88), the evaluation of the resulting expression involves integrals of the type

$$\int_0^T e^{p\tau} e^{-k_r |t-\tau|} d\tau = \frac{2k_r e^{pt}}{p^2 - k_r^2} - \frac{e^{-k_r t}}{p + k_r} + \frac{e^{k_r t} + (p - k_r)T}{p - k_r} \quad (90)$$

Hence

$$\begin{aligned} \frac{1}{\lambda} \phi(t) &= \frac{1}{\lambda} \left[\sum_{m=1}^n A_m e^{-\gamma_m t} + \sum B_m e^{\gamma_m t} \right] = \int_0^T \phi(\tau) K |t-\tau| d\tau \\ &= \sum_{m=1}^n \int_0^T A_m e^{\gamma_m t} \sum_{r=1}^n C_r e^{-k_r |t-\tau|} d\tau + \sum_{m=1}^n \int_0^T B_m e^{-\gamma_m t} \sum C_r e^{-k_r |t-\tau|} d\tau \\ &= -2 \sum_{mr} \frac{k_r A_m C_r}{\gamma_m^2 - k_r^2} e^{-\gamma_m t} - \sum_{mr} \frac{A_m C_r e^{-k_r t}}{-\gamma_m + k_r} - \sum_{mr} A_m C_r \frac{e^{k_r t} - (k_r + \gamma_m)T}{k_r + \gamma_m} \\ &\quad - 2 \sum_{mr} \frac{k_r B_m C_r e^{\gamma_m t}}{\gamma_m^2 - k_r^2} - \sum_{mr} \frac{B_m C_r e^{-k_r t}}{k_r + \gamma_m} - \sum_{mr} \frac{B_m C_r e^{k_r t - (k_r - \gamma_m)T}}{k_r - \gamma_m} \end{aligned} \quad (91)$$

If (91) is to be true for all t , the coefficients of $e^{-\gamma_m t}$ and $e^{\gamma_m t}$ in both sides of (91) must agree for all m and the coefficients of $e^{-k_r t}$ and $e^{k_r t}$ must vanish for all r .

Applying the second condition, one has from (91) Coeff of $e^{k_r t}$ is

$e^{-k_r t}$ and $e^{k_r t}$ in the integrated expression of

$$\int_0^T \phi(\tau) K |t-\tau| d\tau,$$

can be made equal to zero, then it will be possible to obtain the solution of the eigenvalue problem.

When (89) is substituted in (88), the evaluation of the resulting expression involves integrals of the type

$$\int_0^T e^{p\tau} e^{-k_r |t-\tau|} d\tau = \frac{2k_r e^{pt}}{p^2 - k_r^2} - \frac{e^{-k_r t}}{p + k_r} + \frac{e^{k_r t} + (p - k_r)T}{p - k_r} \quad (90)$$

Hence

$$\begin{aligned} \frac{1}{\lambda} \phi(t) &= \frac{1}{\lambda} \left[\sum_{m=1}^n A_m e^{-\gamma_m t} + \sum B_m e^{\gamma_m t} \right] = \int_0^T \phi(\tau) K |t-\tau| d\tau \\ &= \sum_{m=1}^n \int_0^T A_m e^{\gamma_m t} \sum_{r=1}^n C_r e^{-k_r |t-\tau|} d\tau + \sum_{m=1}^n \int_0^T B_m e^{-\gamma_m t} \sum C_r e^{-k_r |t-\tau|} d\tau \\ &= -2 \sum_{mr} \frac{k_r A_m C_r}{\gamma_m^2 - k_r^2} e^{-\gamma_m t} - \sum_{mr} \frac{A_m C_r e^{-k_r t}}{-\gamma_m + k_r} - \sum_{mr} A_m C_r \frac{e^{k_r t} - (k_r + \gamma_m)T}{k_r + \gamma_m} \\ &\quad - 2 \sum_{mr} \frac{k_r B_m C_r e^{\gamma_m t}}{\gamma_m^2 - k_r^2} - \sum_{mr} \frac{B_m C_r e^{-k_r t}}{k_r + \gamma_m} - \sum_{mr} \frac{B_m C_r e^{k_r t - (k_r - \gamma_m)T}}{k_r - \gamma_m} \end{aligned} \quad (91)$$

If (91) is to be true for all t , the coefficients of $e^{-\gamma_m t}$ and $e^{\gamma_m t}$ in both sides of (91) must agree for all m and the coefficients of $e^{-k_r t}$ and $e^{k_r t}$ must vanish for all r .

Applying the second condition, one has from (91) Coeff of $e^{k_r t}$ is

equal to zero for all 'r' or,

$$\sum_m \frac{A_m}{k_r - \gamma_m} + \sum_m \frac{B_m}{k_r + \gamma_m} = 0$$

Coeff of $e^{-k_r t}$ is equal to zero for all 'r', or (92)

$$\sum_m \frac{A_m e^{-\gamma_m T}}{k_r + \gamma_m} + \sum_m \frac{B_m e^{\gamma_m T}}{k_r + \gamma_m} = 0$$

Here the common factor $e^{-k_r t}$ has been canceled from the second equation in (92). Equation (92) is a homogeneous set of '2n' equations for the coefficients A_m and B_m .

Also by equating the coefficients of $e^{-\gamma_m t}$ or $e^{\gamma_m t}$, for each 'm' in both sides of (91), the following is obtained.

$$\frac{1}{\lambda} = -2 \sum_r \frac{k_r C_r}{\gamma_m^2 - k_r^2} \quad r = 1, 2, \dots, n \quad (93)$$

The routine for calculating λ and the unknowns in the expansion of the eigenfunction, viz, A_m , B_m and γ_m is the following.

Noting that Equation (93) is an algebraic equation of 'n'th order for γ_m^2 and that the coefficients of this equation in value λ , the following steps can be outlined.

- 1 Find 'n' roots of r_m^2 , viz $\gamma_1^2, \gamma_2^2 \dots \gamma_n^2$ from (93). Note that the expression for these roots involve ' λ '.
2. Substitute $\gamma_1, \gamma_2, \dots$ etc. in (92). Since (92) is a homogeneous equation for A_m 's and B_m 's, the determinant must be equal to zero. Using $\Delta = 0$ where Δ is the determinant of (92), solve the transcendental equations for λ , yielding an infinite number of solutions $\lambda_1, \lambda_2, \dots$.
- 3 For each λ , say λ_p calculate γ_{mp} ($m = 1, \dots, n$) from the expressions in step 1.
4. Using these γ_{mp} 's solve (92) for A_{mp} 's and B_{mp} 's corresponding to λ_p . Note that the coefficients can be determined only within a multiplicative

factor. The expression for the p^{th} eigenfunction is then

$$\phi_p = \sum_m A_{mp} e^{-\gamma_{mp} t} + \sum_m B_{mp} e^{+\gamma_{mp} t} \quad (94)$$

The difficulty in actually working through the above procedure is encountered, first, in solving for the 'n' roots of λ_m in step 1, since the coefficients involve the unknown λ . Even more difficult is the solution of $\Delta(\lambda) = 0$, since this equation is transcendental. Only approximate solutions can therefore be expected.

Example:

Consider the case $K|t-\tau| = e^{-k|t-\tau|}$, i.e., $n = 1$

Equation (93) is then,

$$\frac{1}{\lambda} + \frac{2k}{\gamma^2 - k^2} = 0$$

or

$$\gamma^2 + 2k\lambda - k^2 = 0 \quad \text{or} \quad \gamma = (k^2 - 2k\lambda)^{\frac{1}{2}} \quad (95)$$

From (92) one has,

$$\left. \begin{aligned} \frac{A}{k-\gamma} + \frac{B}{k+\gamma} &= 0 \\ \frac{Ae^{-\gamma T}}{k+\gamma} + \frac{Be^{\gamma T}}{k-\gamma} &= 0 \end{aligned} \right\} \quad (96)$$

$$\Delta \begin{vmatrix} \frac{1}{k-\gamma} & \frac{1}{k+\gamma} \\ \frac{e^{-\gamma T}}{k+\gamma} & \frac{e^{-\gamma T}}{k-\gamma} \end{vmatrix} = 0$$

or

$$\frac{e^{\gamma T}}{(k-\gamma)^2} = \frac{e^{-\gamma T}}{(k+\gamma)^2} = 0$$

or

$$e^{\gamma t} = \pm \frac{k-\gamma}{k+\gamma} \quad (97)$$

Substituting from (95) in (97)

$$e^{(k^2 - 2k\lambda)^{\frac{1}{2}} T} = \pm \frac{k - (k^2 - 2k\lambda)^{\frac{1}{2}}}{k + (k^2 - 2k\lambda)^{\frac{1}{2}}} \quad (98)$$

Solutions of (98) which is a transcendental equation for λ are the eigenvalues of the problem.

For a particular λ say λ_p ,

$$\gamma_p = (k^2 - 2k\lambda_p)^{\frac{1}{2}}$$

from (95) and from (96)

$$B_p = \frac{k + \gamma_p}{\gamma_p - k} A_p$$

Hence

$$\phi_p = A_p \left[e^{-\gamma_p t} + \frac{k + \gamma_p}{\gamma_p - k} e^{\gamma_p t} \right] \quad 0 < t < T \quad (99)$$

which rearranged reads,

$$\phi_p = \frac{k}{r_p} \sinh \gamma_p t + \cosh \gamma_p t \quad (100)$$

where the constant factor has been dropped.

The same solution was obtained by Youla⁴ although by using a different method of formulation.

6.2 Upper Limit Infinity

It is of interest to consider the case when T , the upper limit of the Integral Equation is replaced by infinity. The formulation presented in this section can still be used but there results considerable simplification of the process of obtaining λ and ϕ . This is discussed in the following

It is to be noticed in the first instance that because the range of the integral equation is from 0 to ∞ , continuous eigenvalues will result

due to the type of kernel considered. In order for the integral

$$\int_0^{\infty} (\sum C_r e^{-k_r t}) (\sum A_r e^{-\gamma_r t} + B e^{\gamma_r t}) dt$$

to converge it is essential that if $\text{Re } \gamma_r = \gamma_{r1} (\gamma_{r1} > 0)$ then $\gamma_{n1} < k_1$, k_r real. It has been assumed that $k_1 < k_2 < \dots < k_n$ and $\gamma_{11} < \gamma_{21} < \dots < \gamma_{n1}$. It is to be noted that since $\gamma_{r1} > 0$, all $-\gamma_r$'s are admissible.

Since the γ 's involve the eigenvalue λ , this sets a limit to the values that λ can take. Consider a simple example in which $k|t-\tau| = e^{-k|t-\tau|}$. Then the integral equation is,

$$\phi(t) = \lambda \int_0^{\infty} (\tau) e^{-k|t-\tau|} d\tau, \quad k \text{ real} \quad (101)$$

Assuming $\phi(\tau)$ to be of the form,

$$\phi(\tau) = A e^{-\gamma \tau} + B e^{\gamma \tau} \quad (102)$$

one gets by substituting (102) in (101),

$$\frac{1}{\lambda} (A e^{\gamma t} + B e^{-\gamma t}) = -2k \frac{A e^{-\gamma t}}{\gamma^2 - k^2} - \frac{A e^{-kt}}{\gamma + k} - 2k \frac{B e^{\gamma t}}{\gamma^2 - k^2} - \frac{B e^{-kt}}{-\gamma + k} \quad (103)$$

Hence

$$\frac{1}{\lambda} = - \frac{2k}{\gamma^2 - k^2} \quad (104)$$

and

$$\frac{A}{\gamma + k} + \frac{B}{-\gamma + k} = 0 \quad (105)$$

from (104)

$$\gamma^2 = k^2 - 2k\lambda \quad \text{or} \quad \gamma = \sqrt{k^2 - 2k\lambda} \quad (106)$$

In (105), A can be arbitrarily chosen and solving for B one gets,

$$B = \frac{-k}{+k} A \quad (107)$$

Note that (105) can be satisfied for all γ by the choice of B/A . Therefore, the eigenvalue equation (105) is satisfied for a continuous range of λ . The solution $\phi(t)$ is then

$$\phi(t) = A \left[e^{-\sqrt{k^2 - 2k\lambda} t} + \frac{\sqrt{k^2 - 2k\lambda + k}}{\sqrt{k^2 + 2k\lambda + k}} e^{\sqrt{k^2 - 2\lambda k} t} \right] \quad (108)$$

Note that if $\text{Re } \lambda < 0$ then $\text{Re}(\gamma) > k$ hence the negative values of $\text{Re}(\lambda)$ are inadmissible.

Equation (101) was solved by the Wiener-Hopf technique and appears as an example in the book by Morse and Feshbach¹.

When the series representation of the kernel contains more than one term say ' p ', then it is found that the problem reduces to finding $(p+1)$ coefficients from ' p ' simultaneous equations. This can be done by choosing one of the coefficients arbitrarily and solving the ' p ' equations.

The eigenvalue spectrum is continuous, hence the allowed values of λ are arbitrary, except for the restriction on $\text{Re}(\lambda)$. This restriction is imposed from the convergence condition, i.e. the restriction on $\text{Re}(\gamma)$, which was discussed earlier.

7. NUMERICAL EXAMPLE

In this Chapter a numerical example is worked out to demonstrate the application of the theory discussed in Chapters 2 and 3. The equation considered is of the form (1)

Let the kernel $K |t-\tau|$ of the form

$$K |t-\tau| = e^{-|t-\tau|} + 2e^{-2|t-\tau|} - e^{-3|t-\tau|}$$

i.e. $k_1 = 1, k_2 = 2, k_3 = 3$

$F(\omega^2)$, the Fourier transform of $K(t)$ is

$$F(\omega^2) = 2 \left[\frac{1}{\omega^2 + 1} + \frac{2}{\omega^2 + 4} - \frac{1}{\omega^2 + 9} \right]$$

a_i 's are obtained from the roots of $F(\omega^2)$, and are

$$a_1 = 0.95837 \qquad a_2 = 3.44949$$

The matrix $[M]$ which corresponds to the case of $T = \infty$ is then,

$$[M] = \begin{bmatrix} 1 & -1.84628 & 0.40825 \\ 1 & -0.64866 & 0.68990 \\ 1 & -0.39345 & 2.22474 \end{bmatrix}$$

The cofactors and the determinant for $[M]$ and $[M']$ calculated using the expressions in Chapters 3 and 4 and verified by direct calculation are,

$\Delta_{11} = -1.17167$	$\Delta'_{11} = -1.28444$
$\Delta_{21} = -3.94688$	$\Delta'_{21} = 4.17186$
$\Delta_{31} = -1.00893$	$\Delta'_{31} = -1.05800$
$\Delta_{12} = -1.53485$	$\Delta'_{12} = -1.53485$
$\Delta_{13} = 0.25522$	$\Delta'_{13} = 0.27322$
$\Delta_{22} = 1.81650$	$\Delta'_{22} = 1.81650$
$\Delta_{23} = -1.45283$	$\Delta'_{23} = -1.51622$
$\Delta_{32} = -0.28165$	$\Delta'_{32} = -0.28165$
$\Delta_{33} = 1.19762$	$\Delta'_{33} = 1.24221$

$$\Delta = 1.76627$$

$$\Delta' = 1.82941$$

The inverse of $[M]$ and $[M']$ constructed from the above data are

$$[M]^{-1} = \begin{bmatrix} -0.66336 & 2.23458 & -0.57122 \\ -0.86898 & 1.02844 & -0.15946 \\ -0.14450 & -0.82254 & 0.67805 \end{bmatrix}$$

$$[M']^{-1} = \begin{bmatrix} -0.70211 & -2.28044 & -0.57833 \\ -0.83899 & 0.99294 & -0.15396 \\ 0.14935 & -0.82880 & 0.67902 \end{bmatrix}$$

Once the inverses $[M]^{-1}$ and $[M']^{-1}$ are known the Integral Equations for both $T = \infty$ and a finite T can be obtained by the methods outlined in Chapters 2 and 3.

8. APPLICATIONS

The Wiener-Hopf Equation for the finite and infinite ranges occur in the problems associated with the theory of linear prediction and the design of optimum filters in the field of Information Theory.

In the field of Electromagnetic Theory, the Wiener-Hopf Equation occurs in numerous problems of wave propagation.

Examples are the problems of diffraction by a slit, of finite width in an infinite plane, or the diffraction by an infinite stack of equispaced metal planes. Several other examples are also scattered around in the literature on wave propagation and boundary value problems.

9. CONCLUSIONS

In this report, methods have been developed for the solution of finite and infinite range Integral Equations of Wiener-Hopf type and the eigenvalue problem associated with the kernel. The assumption in a part of the work is that the kernel $K|t-\tau|$ can be conveniently expressed as a series of exponentials. Iterative methods are presented in another part for the case where the series expansion is inconvenient.

Examples are worked out to illustrate the methods. Applications in the fields of Information Theory and E.M. wave propagation are pointed out. Actual application to a particular problem in field theory viz. the diffraction by a stack of finite-sized metal planes is left for discussion in a future report.

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